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Making Algebra Work: Instructional Strategies That Deepen Student Understanding, Within and Between Algebraic Representations

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Competence in algebra is increasingly recognized as a critical milestone in students' middle and high school years. The transition from arithmetic to algebra is a notoriously difficult one, and improvements in algebra instruction are greatly needed (National Research Council, 2001). Algebra historically has represented students' first sustained exposure to the abstraction and symbolism that makes mathematics powerful (Kieran, 1992); its symbolic procedures enable students to consider relationships, variable quantities, and situations in which change occurs (Fey, 1990). In addition to its central role in the discipline of mathematics, algebra also serves as a critical "gatekeeper" course, in that earning a passing grade has become a de facto requirement for many educational and workplace opportunities. Some have gone so far as to refer to algebra as the new civil right (Moses, 1993). Research shows that students who complete a mathematics course beyond the level of Algebra II more than double the odds of pursuing and completing post-secondary education (Adelman, 1999). Many districts now require completion of an Algebra I course prior to completion of 9th grade (Loveless, 2008).

Regrettably, students' difficulties in algebra have been well documented in national and international assessments (e.g., Beaton et al., 1996; Blume & Heckman, 1997; Lindquist, 1989; Schmidt, McKnight, Cogan, Jakwerth, & Houang, 1999). For example, data from the National Assessment of Educational Progress [NAEP] indicates that many 12th graders can solve only the most simple and routine algebra tasks (Blume & Heckman, 1997).

Challenges Associated with Improving Students' Learning of Algebra

Teachers, schools, and districts that are interested in improving students' learning of algebra face numerous challenges. First, there is not agreement in the practitioner or research communities concerning what algebra is. For some teachers and researchers, algebra is fundamentally about manipulating symbolic expressions and equations and gaining fluency with symbolic procedures. Others conceptualize the central concern of algebra to be the exploration of and representations of functions or, more generally, about relationships between quantities that vary. Recently, a third view of algebra has prioritized algebraic reasoning as being more central than learning formal symbolic procedures for manipulating expressions. Certainly all of these views have merit. But teachers and districts often face choices between these different approaches when selecting curriculum, and little evidence (in terms of research on which, if any, approach is optimal) is available.

Second, educators and researchers also do not agree on what concepts and skills are critical prerequisites to later success in algebra, and the research base for various views on this issue is almost completely lacking. For example, the National Mathematics Advisory Panel (2008) recently focused its attention on three areas that it viewed as prerequisites for algebra: Fluency with arithmetic operations, rational number knowledge (including fractions), and measurement. There is certainly strong intuitive and theoretical support for the importance of these three areas, but there is no research directly linking each to student outcomes. For example, is it the case that students who have fluency with operations on fractions (such as adding fractions with unlike denominators) do better in a subsequent Algebra I course than those who lack such fluency? Research for even basic assumptions such as this one are almost entirely absent. An alternative view about prerequisite knowledge that is held by some researchers and educators suggests that later success in Algebra I can be enhanced by the exploration of symbolic algebra in elementary school. Very interesting and innovative research is being conducted in many elementary schools, indicating that young students are surprisingly capable of doing and understanding algebra concepts and skills that were previously viewed as beyond their capacity. Yet again, little or no research shows that students who engage with and understand symbolic algebra in elementary school subsequently do better in Algebra I courses.

A third challenge facing educators seeking to improve students' performance and understanding in algebra is that there are divergent views on when students should receive instruction on symbolic algebra. It was not that long ago when students uniformly were introduced to formal school algebra in 9th grade, with an Algebra I course. In this organization of the curriculum, middle school was viewed as an opportunity to lay conceptual foundations for later symbolic and abstract study of school algebra. In the 1990s, many educators proposed an alternative -- that the high school curriculum should be more integrated. As a result, topics from the traditional Algebra I course became more dispersed throughout the high school curriculum. And recently, with the release of the National Mathematics Advisory Panel (2008) report and National Council of Teachers of Mathematics [NCTM] Focal Points (National Council of Teachers of Mathematics, 2006), algebra instruction appears to be shifting to middle school, with some topics traditionally associated with the Algebra I course such as linear equation solving receiving instructional emphasis as early as 7th grade in regular track courses. (And similar to the point above, there is no research base to indicate the optimal time and curriculum structure for introducing symbolic algebra.)

Finally, educators are challenged by public misperception of what algebra is and what it is used for. To many laypersons, algebra was centrally concerned with mysterious games played with the last three letters of the alphabet. Furthermore, most adults (even those who had positive experiences with algebra) do not perceive that they use algebra in their daily personal or professional lives. For parents who hold these views on algebra, it is understandable that questions might arise about districts' push toward algebra for all students.

Our Conception of Algebra

Given the above discussion about differing perspectives on what algebra is, it seems important to articulate our view of what algebra is before describing instructional recommendations designed to improve performance and understanding of algebra.

To begin, we consider algebra to be about using tables, graphs, and symbols to explore relationships between quantities. By relationships between quantities, consider the following examples. There is a relationship between my cell phone bill and how many minutes I talk on the phone. Similarly, there is a relationship between the balance of my savings account and how much money I add or subtract each week, the interest rate, and the balance. There is a relationship between the profit we make at a bake sale, and the amount of cookies we sell, and the price of the ingredients. In these and other similar situations, it becomes possible to ask questions about the relationships between various quantities that we can subsequently explore with tables, graphs, and symbols. Collectively, we refer to tables, graphs, and symbols as representations. Representations allow us to explore, generalize, predict, and analyze features of situations where quantities vary.

If one views algebra as fundamentally involving the use of representations to explore relationships between varying quantities, then understanding in algebra can be considered to consist of two complimentary capacities, which we refer to as *between* and *within* representational fluency. The first concerns the ability to operate fluently between and across multiple representations, while the second is about facility within each individual representation. We elaborate on these two capacities, below.

With respect to the first, *between* representational fluency, this capacity means that students can analyze situations using graphs, tables, and symbols and subsequently make connections between these representations. Making these connections is a critical part of what we (and many others) think it means to understand algebra. Revisiting the cell phone bill example from above (see Figure 1), a common billing arrangement is that one pays a flat rate for a set number of minutes, and then an extra fee for each minute that is used over this set limit. Thinking *between* representations means understanding what the extra charge-per-minute looks like on a graph—it is the slope. Similarly, the ability to move fluently between representations involves understanding what the per minute charge look like on a table. It is not enough that a student could take a given situation and generate one representation; understanding in algebra means being able to use multiple representations and then to connect between them. The student who can talk intelligently about the extra charge (meaning the slope) and what this feature of the situation looks like on a graph and table understands much more about linear relationships than a student who is only able to produce one representation for this situation.

In addition to between representational fluency, a related competency in algebra is *within* representational fluency. Certainly students need to know the concepts and skills that are related to working within a single representation, but they also need to know multiple strategies within that representation, including the ability to select the most appropriate strategies for a given problem. Elsewhere we refer to this within-representational fluency as *flexibility,* or *strategy flexibility* (Star & Seifert, 2006; Star & Rittle-Johnson, 2008; Rittle-Johnson & Star, 2007). As an example, consider strategies within a single representation (graphical) for graphing a line. There are many ways to graph a line. In particular, we can plot any two points; we can use the slope and the *y*-intercept; or we can use two special points (the *x*- and *y*-intercepts). The student who understands what he or she is doing in algebra with respect to graphing lines knows multiple ways to graph a line, and chooses to graph differently depending on the particulars of the lines to be graphed and/or the problem-solving situation. For example, a flexible student might choose to graph $3x + 2y = 12$ differently than $y = \frac{2}{3}x - 4$ (see Figure 2); in the former, the *x*- and *y*-intercepts are easily identified ((4,0) and (0,6)) and can be plotted; in the latter, the slope and *y*-intercept are easily identified ($\frac{2}{3}$ and (0,-4)) and can be use to graph this line. Similarly and within the symbolic representation, there are many ways to solve equations, to simplify exponential expressions, to solve linear systems, etc. Understanding algebra means knowing multiple ways to solve equations, and choosing a particular solution for a given a problem because it is the best one.

Instructionally we believe that it is not a good idea to teach students one and only one way to approach a type of mathematics problems. Teachers may think that they are doing students a favor by focusing on only one strategy, believing that they are making things easier by only focusing on one way. Such a teacher might preface his/her instruction on a single strategy by saying, “This method is all you need to remember; this method is the one way to solve this type of problem." But if students only know one way to solve a problem, and if they forget that way, or if they see a problem that they do not recognize, they are stuck. Alternatively, if students have a more robust knowledge within a single representation and they can approach a given problem in multiple ways (in other words, *within* representational fluency), they are better prepared to tackle both familiar and unfamiliar problems. Such flexibility is a key component of what it means to understand in algebra.

Using Comparison to Improve Students' Learning of Algebra

If our goal is to improve students' understanding of algebra, including both between and within representational fluency, one important tool that teachers have to help students learn multiple approaches is comparison. For at least the past 20 years, a central tenet of effective instruction in mathematics has been that students benefit from sharing and comparing solution methods (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). Case studies of mathematics teachers indicate that expert teachers are more likely to encourage students to actively compare solution methods. Furthermore, teachers in countries where student performance in math is particularly high, such as Japan and Hong Kong, frequently have students compare multiple solution methods during their lessons (Stigler & Hiebert, 1999). This emphasis on sharing and comparing solution methods was formalized in the National Council of Teachers of Mathematics *Standards* (1989, 2000).

Comparison helps focus learners' attention on critical features of examples. The utility of comparison can be seen intuitively by considering an example from everyday life. Suppose we were interested in purchasing a digital camera from an online electronics store. From the many options available, how could we select which camera to purchase? Before going online, from many possible cameras in our price range, imagine that we narrowed our choice down to two cameras. One possible strategy would be to view the information about each camera separately, reading the specifications, reviews, and price information for each camera individually. However, reading the long lists of dozens of features associated with each camera separately would make it very difficult for us to notice what features of the cameras were similar and what features were different. But now imagine instead if we could compare the same information by viewing the features of the two cameras under consideration at the *same time*, side by side. A quick glance at a comparison chart could allow us to easily distinguish whether the cameras were the same or different for the features that were important to us. Comparing the cameras side by side could help us to easily identify the salient similarities and differences among the products (which is probably why many online retailers provide a tool for consumers to compare products side by side).

The power of comparison as seen while shopping for digital cameras could also be utilized in the mathematics classroom to help students learn multiple strategies. However, some common instructional practices used by many mathematics teachers may cause teachers to miss opportunities to realize the benefits of comparison with their students. Suppose an algebra teacher, Mr. S, has the goal of helping his students to become more flexible learners; in particular, he wants the students to develop *within* representational fluency, or multiple strategies for solving a given problem in a single representation. In pursuit of this goal, he decides to show students several examples solved using several solution methods. Mr. S begins his lesson by working through his first example problem on the board. He explains the solution method to the class, then erases the board. Next, he writes another, similar example on the board, and solves it using the same solution method he used for the first example. “Now,” he tells the class, “I will show you a third example, which is a little different from the first two.” He erases the board, then puts up the third example, which he solves using a different solution method from the one that was used for examples one and two.

While Mr. S’s intentions in introducing several types of problems and strategies to his students were sound, the method he utilized in the above example caused him to miss an opportunity to use the power of comparison for student learning. Despite his good intentions, Mr. S may not realize his goal of having students become flexible and knowledgeable about multiple solution strategies, because he has made it very difficult for students to compare and contrast multiple approaches. In the example above, Mr. S assumes that his students notice the ways in which the third example is different from the first two, even though only one problem was visible on the board to students at a time. Mr. S also assumes that his students notice the ways in which the method used to solve the third problem is different from the method he used to solve the first two problems, even though only one method was on the board at a time. And finally, Mr. S assumes that his students can notice and distinguish the features of the third problem that led Mr. S to decide that an alternative solution method would be a better strategy for approaching this problem. This is a critical distinction, and one that would be quite difficult for students to make when they can only see one problem solved on the board at a time. In essence, Mr. S is not effectively facilitating comparison in his classroom, because his practice of erasing the board after solving each problem hinders his students' ability to draw comparisons among the problems and solutions strategies he has presented. As with the digital cameras, it is very difficult to notice the salient similarities and differences among multiple items unless their features are compared side by side.

Imagine that if, instead of erasing the board between the examples, Mr. S had manipulated the space on the board such that he solved all of the worked examples side by side. With the solution methods presented immediately adjacent to one another, students could quickly and easily see the similarities and differences in the problems and their solutions. The juxtaposition of the worked examples side-by-side would have allowed Ms. S to physically point out to students the differences between the problems, providing students with a visual aid to support comparisons of the multiple problems. Indeed, research indicates that students who study worked examples side by side, with prompts to compare and contrast the examples, become better problem solvers and develop greater flexibility than students who study the same examples listed one at a time (Rittle-Johnson & Star, 2007). With a relatively small change in his instructional practice, presenting problems on the board side by side rather than erasing after each example, Mr. S would be more likely to improve students' knowledge of multiple strategies for solving problems. By presenting problems side-by-side, Mr. S could more effectively facilitate conversations with students involving comparing and contrasting multiple problems and solution methods.

The importance of comparison for mathematics instruction is confirmed in international studies of mathematics teachers. Researchers evaluated the use of comparison in typical math classrooms in the United States, Japan, and Hong Kong and found that expert teachers in all three countries frequently used comparison as a tool for teaching math. They compared new math concepts to ideas that were already familiar to students, carefully placing examples side-by-side and using hand gestures to highlight similarities and differences (Richland, Zur, & Holyoak, 2007). According to Richland, comparison "allows students to use commonalities between mathematical representations to help understand new problems or concepts, thereby contributing to integral components of mathematical proficiency" (Richland et al., 2007, p. 1128).

Three Instructional Practices Utilizing Comparison

Given the benefits of comparison, how can teachers modify existing instructional practices to utilize the power of comparison to help develop students' understanding in algebra, including both between and within representational fluency?

First, and as suggested above in the Mr. S example, research on comparison indicates to-be-compared solution strategies must be presented to students side-by-side, rather than sequentially. Side-by-side placement allows for more direct comparison of solution strategies and facilitates the identification of similarities and differences between strategies. A side-by-side comparison helps students notice and remember the features that are important to each or both solution strategies (Rittle-Johnson & Star, 2007). Using common labels in the examples should also help students notice the similarities and differences.

It is important to note that it was *not* merely exposure to multiple strategies that helped students become better equation solvers in past research. Rather, it was the side-by-side placement of the multiple strategies, combined with opportunities for comparison conversations, that led to the gains experienced by comparison group students. Thus, a second practice that utilizes comparison that has been found to improve algebra learning is for teachers to engage students in comparison conversations. Discussion of and comparison of multiple strategies helps students justify why a particular solution strategy or solution step is acceptable and helps students make sense of why certain strategies are more efficient than others for particular problems (Silver et al., 2005). Teachers can help guide comparison conversations to ensure that students are able to make connections among strategies that they would not always be able to make on their own. In addition, Rittle-Johnson and Star (2007) showed that comparison conversations could also happen in student pairs; student discussion pairs were able to work together to identify problem features, and evaluate and compare the accuracy and efficiency of different solution strategies. Engaging in discussions seemed to enable students to more readily accept nonstandard strategies.

The final recommended practice is to provide students with the opportunity to generate multiple solution methods to the same problem, either by investigating multiple solutions of the same equation or by creating new equations to solve by a given method. In general, knowledge of multiple solution strategies seems to help students more readily consider efficiency and accuracy when solving problems. Additionally, by generating multiple solutions, students are encouraged to move away from using a single strategy and consider other, possibly better strategies that work for the problem (Star & Rittle-Johnson, 2008; Star & Seifert, 2006).

Implementation of Instructional Practices

The research on comparison practices alluded to above has focused on whether comparison improves students' learning of algebra. However, there is also emerging evidence on the ways that these practices can impact teachers’ practices. In a recent study, algebra teachers participated in professional development designed to introduce them to these three comparison practices (Yakes & Star, 2008). Twenty-four middle and high school mathematics teachers in California participated in a two-week institute that focused on algebraic reasoning and pedagogical strategies for use in algebra classrooms, particularly the comparison practices described above. After the professional development course, teachers also were given academic year follow-up consisting of face-to-face meetings and an online support community.

Analysis of teachers' experiences in the professional development course indicates that when teachers are presented with techniques for effective use of comparison, their own understanding of multiple solution methods is reinforced. In addition, teachers began to question why they relied exclusively on one familiar method over others that are equally effective and perhaps more efficient and drew new connections between problem solving methods. Finally, as a result of experiencing instructional use of comparison, teachers began to see value in teaching for flexibility and appeared to change their own teaching practices.

Teachers' enthusiasm for the comparison practices was reflected in selected responses that they provided on a survey administered after the professional development institute. One teacher noted that, "If students look at several ways of doing the same problem, they can start to generalize what’s really going on.” Similarly, another teacher indicated that, "I know that when I was learning math, I often fell back on one way of solving a problem. I think this did not allow for a better understanding of the topic because I was so focused on one solution method. This one-way method put up a sort of roadblock in my understanding."

Conclusion

To summarize, success in algebra is widely recognized as critical to students’ future success in later mathematics courses and in post-secondary education. Educators who are interested in improving students’ performance face numerous challenges, including a lack of agreement about what algebra is, minimal research on which prerequisite concepts and skills predict later success in algebra, public misperceptions of the role of algebra in the workplace, and lack of consensus or research on when students should optimally be exposed to symbolic algebra. Our vision of algebra involves using representations to explore relationships between varying quantities, and we articulate a view of competence in algebra that involves both between and within representational fluency. We identify the important role that comparison plays in students’ learning of algebra, and we describe three instructional practices that our own research has identified as effective in helping students harness the power of comparison to improve their learning of algebra, both between and within representations.

References

Adelman, C. (1999). *Answers in the toolbox: Academic intensity, attendance patterns, and bachelor's degree attainment*. Washington, D.C.: U.S. Department of Education Office of Educational Research and Improvement.

Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzales, E. J., Kelly, D. L., & Smith, T. A. (1996). *Mathematics achievement in the middle years: IEA's third international mathematics and science study.* Boston: Center for the Study of Testing, Evaluation, and Educational Policy, Boston College.

Blume, G. W., & Heckman, D. S. (1997). What do students know about algebra and functions? In P. A. Kenney & E. A. Silver (Eds.), *Results from the sixth mathematics assessment* (pp. 225-277). Reston, VA: National Council of Teachers of Mathematics.

Fey, J. T. (1990). Quantity. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 61-94). Washington, D.C.: National Academy Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Simon & Schuster.

Lindquist, M. M. (Ed.). (1989). *Results from the fourth mathematics assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Loveless, T. (2008). *The misplaced math student: Lost in eighth-grade algebra*. Washington, DC: Brown Center on Education Policy, Brookings Institution.

Moses, R. (1993). *Algebra - the new "civil right"*. Paper presented at the SUMMAC II conference, Cambridge, MA.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va: Author.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, Va: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum Focal Points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the national mathematics advisory panel.* Washington, DC: U.S. Department of Education.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Richland, L. E., Zur, O., & Holyoak, K. J. (2007). Cognitive supports for analogies in the mathematics classroom. *Science, 316*, 1128-1129.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561-574.

Schmidt, W. H., McKnight, C. C., Cogan, L. S., Jakwerth, P. M., & Houang, R. T. (1999). *Facing the consequences: Using TIMMS for a closer look at U.S. mathematics and science education.* Dordrecht: Kluwer.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24*, 287-301.

Star, J. R., & Rittle-Johnson, B. (2008). The development of flexible knowledge: The case of equation solving. *Learning and Instruction, 18*(6), 565-579.

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*(280-300).

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Yakes, C., & Star, J. R. (2008). Using comparison to develop teachers' flexibility in algebra. *Manuscript under review*.